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SOME NEW RATIOS OF CONIC CURVES.

By ALAN S. HAWKESWORTH.

[Concluded from January Number.]

Theorem 9. The focal chords through corresponding points [p and P] upon reciprocal curves determine at their other extremities two other corresponding points [q and Q]; and are thus corresponding chords [Figure 2].

Let p on the ellipse, and P on the hyperbola be the corresponding points, determined by the common radius $S'pEP$. And let pfq and PSQ be the respective focal chords. Then will qQ be also corresponding points, and have the determining radius $S'qFQ$ in common.

For, angles pfS' and PSd are equal [Theorem 6]; and hence also angles qfS' and QSd; or qfa and QSA; which angles must therefore subtend corresponding points [Theorem 6, Corollary 1], upon a common determining radius $S'qFQ$. And similarly, if $p'fq'$ and $P'Q'S$ be the corresponding focal chords; they are determined by the common radius $S'q'Q'$, and diameter $p'S'P'$ [Figure 3]. While, if focal chords through S' are chosen, still more obviously is this true; for the said focal chords and their determining radius or diameter now coincide.

Corollary 1. If lines Ef and Ff, ES and FS be bisected in rr' and RR' respectively, by the several auxiliary circles [Theorem 2], then these points $rr'R'R$ form a rhomboid. For rR and $r'R'$ are fixed, in both magnitude and direction, being parallel to, and one half of fS [Theorem 6, Corollary 3]; while EF, the director chord, which is similarly parallel to, and double of rr', and RR', is clearly variable, both in length and direction.

Theorem 10. The three tangents of the three corresponding points upon

the two reciprocal curves and their common director circle meet each other upon the common directrix. So that the six tangents of two corresponding focal chords, and of the director chord of their two determining radii, will meet in one such point upon the common directrix [Figure 2].

Let *pq* and *PQ* be the two corresponding focal chords; and *EF* their common director circle, between *E* and *F*, the extremities of their two determining radii *S′pEP* and *S′qFQ*.

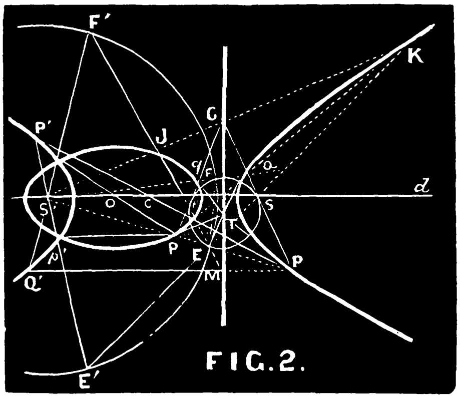

FIG. 2.

Draw, say *PT* and *QT*, the tangents of the focal chord *PSQ* of the hyperbola; meeting in *T* upon the directrix. Join *TE*, *Tp*, *Tq*, *TF*, *Tf*, and *TS*. Then will *Tq* and *Tp* be the tangents to the focal chord *pfq* of the ellipse; and *TE*, *TF* the two tangents at *E* and *F* to the director circle.

For angles *TPE* and *TPS* are equal; sides *PE* and *PS* are equal; and *PT* is common; so that triangle *PET* is equal in all respects to the right angled triangle *PST*. And thus *TE* is both the tangent at *E* to the director circle, and is equal to *TS*. Similarly, *TF* is its tangent at *F*, and is equal to *TS*. Now *fX* equals *XS* [Theorem 7]; and *XT* being a perpendicular, *fT* equals *ST*; and therefore also *TE* and *TF*. Hence the two triangles *pET* and *pfT* are equal in all respects; for they have three sides of each equal; *pE* to *pf*; *ET* to *Tf*; and *pT* common. So that *pET* and *pfT* are right angles; and *pT* bisects externally the angle *S′pf*; and is thus the tangent at *p* to the ellipse. Similarly, *qT* is its tangent at *q*. And since the tangents of a focal chord, in any conic curve, meet each other upon the directrix; therefore *T* is the meeting point of the six tangents *PT*, *QT*, *ET*, and *FT*, *qT* and *pT*.

Were *P′Q′S* and *p′fq′* the focal chords chosen [Figure 3], with their director chord *JK*; then in like manner it can be shown that their respective tangents meet in one and the same point upon the common directrix.

Corollary 1. But, if it be focal chords through *S′*—as *e. g.* *S′pP*—with their resultant director chord identical with the common determining diameter [*S′E*]; then, while the three tangents at *p*, *P*, and *E* meet, as we have seen, in point *T* on the directrix; yet this same point cannot now be also the meeting point of the other three tangents of ellipse, hyperbola, and circle. For, first, the two tangents of a focal chord through *S′* evidently meet, not upon *XT*, the common directrix of foci *f* and *S*, but upon the respective directrix of *S′*; while, secondly, the tangent to the circle, on the opposite extremity of the director chord diameter through *E*, must be parallel to that at *E*. And thus the second set of three tangents must meet in a second point *T′* upon the common directrix *XTT′*.

Corollary 2. Point *T*, the meeting point of the tangents of the focal chords through *f* and *S*, is clearly also the intersection of the directrix *TX* by

the radius which perpendicularly bisects the director chord; even as it is also its intersection by the semi-diameter of ellipse or hyperbola, respectively bisecting the focal chords.

Corollary 3. If, then, with this point T as center, and radius Tf or TS, a circle be described [Figure 2], it will ever circumscribe the simple quadrangle $fESF$, formed by joining the respective foci f and S to the ends of the director chord.

While, on the other hand, it will ever be circumscribed by the quadrilateral $pPQq$; formed by joining the extremities of the two focal chords; and tangential to it at points $fESF$, where the major axis, and director chord, respectively, cut the sides pq, pP, PQ, and qQ; which points evidently coincide with the four corners of the inscribed quadrangle.

And furthermore, this circle will be either inscribed, or escribed to the said quadrilateral $pPQq$, according as PQ, the focal chord of the hyperbola, cuts branch S alone, or both the branches S and S'; thus causing the quadrilateral to be either simple—*e. g.* $pPQq$ [Figure 2]—; or re-entrant—as *e. g.* $p'P'Q'q'$ [Figure 3].

If, however, the chosen focal chords be through S'; with resultant double points T and T', then no circle can be drawn, since all the eight points, corresponding to $pqPQEFf$ and S, now lie collinearly upon the common determining diameter; which is also both focal and director chords.

Theorem 11. If h, upon the major axi, be the point of intersection by the director chord EF, and thus a summit of the quadrangle $fESF$ inscribed to the circle, whose radius is Tf or TS; it will also be the point of intersection of two diagonals pQ and qP of the quadrilateral $pPQq$, circumscribing that same circle, and tangential to it at points $fESF$ [Figure 2].

For the four points $fESF$, being thus common to all three figures—*i. e.* the circle about T, its inscribed quadrangle, and its circumscribing quadrilateral—they must therefore possess in common the self-conjugate triangle hJK.

Corollary 1. Thus J and K are also points in common, being both the other two summits of the inscribed quadrangle, and also the other two intersections of diagonals PqJ, pQK, and $S'JK$, to the circumscribing quadrilateral $pPQq$. So that lines Ef, SF, and Pq are ever concurrent; as also ES, fF, and pQ; and concurrent, too, on a common radius $S'JK$.

Theorem 12. Therefore, even as the fixed points f and S of the two reciprocal curves harmonically divide the diameter [$2S'D$] of their common director circle; so also do any two corresponding points upon the curves harmonically divide the common radius or diameter determining them.

For, taking the circumscribing quadrilateral $pPQq$ [Figure 2], and considering it also as a quadrangle, then point h, where the director chord cuts the major axi [Theorem 11] will not only be the internal intersection of the diagonals pQ and qP; but also one of its three summits, of which the other two are S' and G. While similarly, taking the inscribed quadrangle $fESF$, and considering it also as a quadrilateral, since its three summits coincide with the three in-

tersections of the diagonals of $pPQq$ [Theorem 11, Corollary 1]; so conversely, must the intersections of its three diagonals—*i. e.* S', G, and h—coincide with the three summits of $pPQq$.

And thus the pencil ray of Gh must pass through F and E. So that $S'p$: $S'P = Ep:EP$; and $S'q:S'Q = Fq:FQ$, etc. While, since corresponding focal chords can be drawn through any two corresponding points—such as $p'q'$ and $P'Q'$; or $p'S'P'$—we can in like manner prove that $S'p':S'P' = Jp':JP'$ [Figure 3].

Theorem 13. If pm and PM be the perpendiculars from the reciprocal points p and P respectively, to their common directrix; and E be the extremity of their common determining radius $S'pEP$; then will lines EM and Em pass through f and S, the fixed points [Figure 2].

For $pE:pm = fp:pm = fa:aX = S'f:a'a = S'f:S'D = S'D:S'S = S'E:S'S$; while pm is parallel to $S'S$; and thus angles Epm and $ES'S$ are equal. Therefore Epm and $ES'S$ are similar triangles; and EmS one right line. And in a like manner, $PM:PE = PM:SP = AX:SA = A'A:S'S = S'D:S'S = S'f:S'D = S'f:S'E$; while angles MPE and $ES'f$ are equal. Therefore MEP and fES' are also similar triangles; angles MEP and fES' equal; and thus MEf one right line.

If Q' be taken on the other branch of the hyperbola, with $Q'S'q'F'$ for its determining diameter; and $Q'M'$, and say $q'm'$ for its perpendiculars to the directrix; then similarly $F'fM$ will be one right line; and also $Fm'S$.

Corollary 1. pEm and PME, like $S'ES$ and $S'fE$, are similar triangles; and likewise $q'F'm'$ and $Q'MF'$; $S'F'S$ and $S'fF'$.

Theorem 14. The corresponding chords of reciprocal curves, and their common director chord, all three meet in one point upon their common directrix [Figures 2 and 3].

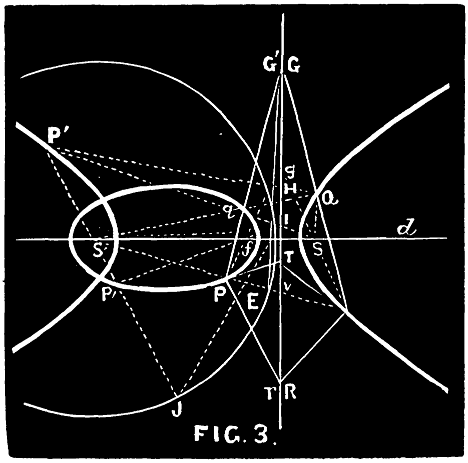

FIG. 3.

First, let the reciprocal focal chords pq and PQ, and their director chord EF, determined by the two radii $S'pEP$ and $S'qFQ$, be the three chords in question [Figure 2], and let pq cut the directrix in, say g; and PQ cut it in G; we will first prove g and G to be identical.

For angles qfa, or gfX, and QSA, or GSX, are equal [Theorem 6]; while fX equals SX [Theorem 7]. So that the right angled triangles fXg and SXG are equal in all respects; Xg to XG, and thus g and G are the same point. Therefore the reciprocal focal chords pqG and PQG meet in the same point G upon the directrix.

Next, G being thus a summit; S' another summit; and $S'pEP$, $S'qFQ$, harmonic ranges upon $pPQq$ considered as a quadrangle [Theorem 12], it follows that GS', Gqp, GFE, and GQP form a harmonic pencil. So that the director chord EF meets its reciprocal focal chords pq and PQ also in G upon the directrix.

And similarly, if $p'fq'I$ and $P'IQ'S$ be the two focal chords chosen [Figure 3], with their common director chord JK, determined by radius $S'q'KQ'$, and diameter $P'S'p'J$; for all three can in like manner be shown to meet in I upon the directrix. While if focal chords through S' be the ones selected, still more evidently do they, and their director chord, meet the directrix in one point; all three chords being coincident with each other, and with the determining diameter.

Secondly, let two non-focal reciprocal chords be chosen, such as pq and PQ [Figure 3]; or $p'q$ and $P'Q$. Then will they, and their common director chord, EF, or JF, as the case may be, still all three meet the directrix in one point G, or H. For, let chord pq cut the directrix in, say, G', and PQ cut it in G; then G and G' are identical. Join points pf, qf, PS, QS, $G'f$, and GS; and let the corresponding focal chords through p and P cut the directrix, as we have just seen, in one and the same point g.

Then, by a well known theorem, $G'f$ externally bisects the focal angle pfq of chord pqG'; and similarly, GS externally bisects angle PSQ. But these angles pfq and PSQ are equal, since $pfS' = PSd$; and $qfS' = QSd$ [Theorem 6]; and thus the halves of their supplements, $G'fg$ and GSg, are also equal, even as also are gfX and gSX. Therefore the right angled triangles $G'fX$ and GSX are equal in all respects; with $G'X$ equal to GX; and thus G' and G are the same point.

Taking $pPQq$ as a quadrangle, G is thus one of its three summits, S' another summit; and $S'pEP$, $S'qFQ$ harmonic ranges upon it; so that GS', Gqp, GFE, and GQP form a harmonic pencil, with ray GFE passing through the third summit; which is also the intersection of diagonals pQ and Pq. And therefore the two non-focal reciprocal chords pqG and PQG with their director chord EF, all three meet in G upon the directrix.

While, if $p'qH$ and $P'HQ$ be the two reciprocal non-focal chords chosen, with their director chord JFH [Figure 3]; in like manner it can be proved that they all three meet the directrix in H. For this is the meeting point of the external and internal bisectors, Hf and HS, of the focal angles $p'fq$ and $P'SQ$, respectively; since the focal chords through p' and P' meet, as we have seen, in I upon the directrix; and the focal angle $p'fq$ is equal to the supplement of focal angle $P'SQ$; $p'fS'$ being equal to $P'SS'$; and qfS' to QSd [Theorem 6], and thus the right angled triangles HXf and HXS are equal in all respects; and chords $p'qH$ and $P'HQ$ meet in the same point H upon the directrix. Which is also a summit of the re-entrant quadrangle $P'Qqp'$; another summit being S'; $S'qFQ$ and $P'S'p'J$ harmonic ranges upon it [Theorem 12]; and thus HS', Hqp', HFJ, and $P'HQ$ an harmonic pencil. So that the reciprocal non-focal chords $p'qH$ and $P'HQ$, with their common director chord JFH, all three meet in one and the same point H upon the common directrix. While the external intersection of the lines through $P'q$ and $p'Q$, since this is the third summit to the re-entrant quadrangle $P'Qqp'$, must lie on the director chord HFJ, since it is a pencil ray; even as the internal summit h [Theorems 11 and 12. Figure 2] lay upon the director chord and pencil ray EF.

Corollary 1. Conversely, then, chords through any two corresponding

points, such as pP, or $p'P'$ [Figure 3], determined by a common radius $S'pEP$, or diameter $P'S'p'J$; and meeting the directrix in the same point G, or H, etc., are thereby corresponding chords, with two other corresponding points qQ, etc. While EFG, or JFH, etc., will be their director chord, cutting the director circle again in F, etc., the extremity of the common radius $S'qFQ$, or diameter, determining these two new points.

Theorem 15. Of two such reciprocal chords, if one pq', or PP' passes through the center [O or C] of its curve, then the other PQ', or pp', as the case may be, is thereby parallel to the major axis, and thus meets its fellow pq', at M, or m, its perpendicular intersection with the directrix. While $F'EM$, or $E'Em$, the director chord in question, has become collinear with fE and fF', or SE and SE', the two lines joining the extremities of the determining radius and diameter to f or S, the fixed point of that chord which passes through the center [Figure 2].

For, let chord pq' of the ellipse pass through its center O; and chord PMQ' be drawn in the hyperbola parallel to the major axis, thus cutting the directrix perpendicularly in M; and being the directrix distances of P and Q'. Then, both ellipse and hyperbola being symmetrical about their respective major and minor axi; angles $q'fS'=pS'f=PS'S=Q'SS'$; and thus $q'Q'$ are corresponding points, lying on the common determining diameter $Q'S'q'F'$ [Theorem 6]; while pq' and PQ' are reciprocal chords, with M for their meeting point on the directrix [Theorem 14].

And, in like manner, if $PmCP'$ be drawn through the center of the hyperbola, and $p'pm$ be parallel to the major axis of the ellipse, $P'SS'=PS'S=pS'f=p'fS'$; so that pp' is the reciprocal chord to PP', and meets it at m on the directrix, the common foot of the directrix distances pm and $p'm$.

Next, the director chord $F'EM$ is collinear with both fE and fF'. For $pf=pE$, and $S'F'=S'E$; so that fpE and $F'S'E$ are isosceles triangles, with sides $S'E$ and pE collinear, and angles pfE, pEf, and $S'F'E$ equal. And thus their bases $F'E$ and fE are also collinear; and coincident with $F'EM$ the director chord [Theorem 13]. Similarly, if $PmCP'$ and mpp' be the two chords in question, with $E'EM$ for their director chord; SPE and $E'S'E$ are isosceles triangles, with sides PE and $S'E$ collinear; while $E'Em$, being Em produced, must pass through S [Theorem 13]. And thus their bases SE and $E'E$ are also collinear with each other, and with $E'Em$ this director chord.

Corollary 1. Therefore, in such a case, $pfqG$ is parallel to $Q'S'q'F'$, or $PSQG$ to $P'S'p'E'$.

Corollary 2. Obviously, if pP lie on a common radius $S'pEP$, then $q'Q'$ or $p'P'$ must lie on a diameter; and conversely.

Corollary 3. Points p' and q', or P' and Q', are also symmetric, their chords $p'q'$ and $P'Q'$ being perpendicular to the major axis. And thus $P'q'$ and $Q'p'$ are equally inclined to the axi; and likewise $P'S'p'E'$ and $Q'S'q'F'$.

Theorem 16. The normals of corresponding points meet upon the common directrix [Figure 3]. Let pP be the two corresponding points; pr and PR their

respective normals, cutting the directrix in say r and R; $S'pEvP$ their determining radius, meeting the directrix in v; and finally pT and PT their respective tangents, meeting each other in T upon the directrix [Theorem 10]. Let $pfp''g$ and $PSP''g$ be the focal chords through p and P, meeting each other in g upon the directrix [Theorem 14], and determining the other two corresponding points p'' and P'' [Theorem 9].

Then, since pr and pT internally and externally bisect the focal angle $S'pf$, and thus also vpg; therefore $rv:rg=Tv:Tg$ [Euclid VI, 3 and A]. And similarly, $S'P$S, or vPg is internally and externally bisected by PT and PR; so that $Rv:Rg=Tv:Tg$. Therefore, since g, T, and v are identical points in both harmonic ranges, points r and R are also identical.

Theorem 17. The normals of the corresponding focal chords, pp'' and PP'' respectively, meet upon the radius passing through T, the common intersection of their tangents with the directrix, and bisecting their common director chord [Theorem 10, Corollary 2].

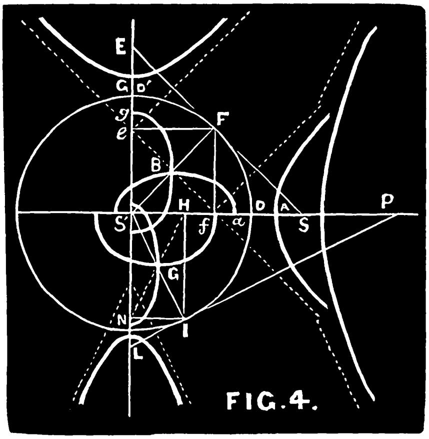

FIG. 4.

For, if such radius, say $S'r'n'TN'R'$ be drawn, meeting the focal chords pp'' and PP'' through p and P in n' and N' respectively; and the normals of p and P in r' and R'. Then, since pr', and pT, PT, and PR', internally and externally bisect the focal angles $S'pf$ and $S'PS$, or $S'pn'$ and $S'PN'$; therefore $S'T:Tn'=r'S':r'n'$; and $S'T:TN'=R'S':R'N'$. While, if the normals of p'' and P'' cut the radius $S'r'n'TN'R'$ in, say, r'' and R'' respectively; then similarly $S'T:Tn'=r''S':r''n'$, and $S'T:TN'=R''S:R''N'$. And S', T, n' and N' being identical points upon the four harmonic ranges $S'r'n'T$, $S'r''n'T$, $S'TN'R'$, and $S'TN'R''$; therefore r' and r'' are also identical points; and likewise R' and R''.

And thus the normals of pp'' meet in r'; and those of PP'' in R', upon the radius $S'r'n'TN'R'$ through T, the common intersection of their four tangents with the directrix; which radius also bisects the common director chord of the reciprocal focal chords.

The foregoing theorems will thus enable us to determine the *Reciprocal* and *Anacyclic* forms to any given conic curve. For the passage of our fixed point, from being the center of its director circle, to a position at infinity, external to said circle, as the focus of a degenerated right line hyperbola, this process, I say, can be now conceived as part of a simultaneous four-fold movement. Since at any point in this course—say, for example, at f [Figure 4]—its generated curve, in this case an ellipse, is balanced and answered by a reciprocal curve, in this case a hyperbola; reverse and opposite in every corresponding point, and gener-

ated by a second fixed point [*e. g.* S], harmonic to the first; rushing from the polar of the center at infinity to meet it on the circumference; and sinking ultimately into the center from whence the original fixed point arose, as that point passes at last to the infinity from whence sprang its fellow. The medial and pivotal conic form in this cycle, as previously stated, being the parabola.

But in addition to these two greater reciprocals, there are also generated two *anacyclic* curves, if we may so name them, or minor reciprocals, in the elliptic and hyperbolic cycles, respectively.

For just as the sine and tangent of the same point [*e. g.* F, or I, Figure 4] on our director circle gave us, on the major axis, the harmonic fixed points of a reciprocal ellipse and hyperbola; so also will the cosine and cotangent of this same point further determine, on the minor axis, two more harmonic fixed points, which will generate a second pair of reciprocal curves—ellipse and hyperbola—anacyclic, and opposite, in many ways, each to its fellow of like form in the first pair. In the hyperbola this anacyclic curve will be a conjugate hyperbola; conjugate, however, not with respect to its position, or magnitude; but solely in regards to its eccentric ratio, or shape. For the common radius [*e. g.* $S'F$, or $S'I$], whose tangent and cotangent pass through the two fixed points generating the said anacyclic hyperbolas, is thereby parallel to an asymptote of each [Theorem 4]; and the said asymptotes, being thus themselves parallel, and so inclusive of supplementary angles with their fellows, must belong to hyperbolas conjugate in ratio. Nevertheless, since the asymptotes are not coincident, but lie on opposite sides of the common determining radius, their respective hyperbolas are not conjugates with respect to position. Nor, again, are they with reference to their magnitude; except, indeed, they be equilaterals; since having a common director circle, their major axi are equal.

The hyperbolic anacyclics are thus conjugates in ratio. And although the same does not hold true of their reciprocal anacyclic ellipses, since the conjugate to an ellipse is never other than itself; yet nevertheless, as the reciprocals of conjugates, they will hold most important relations to each other.

Thus, for example, the right line fe, or HK, joining the anacyclic fixed points, will intersect their common determining radius $S'F$, or $S'I$, on a common extremity, B, or b, of the respective minor axi. Since, joining the fixed points, it is necessarily the bisecting and bisected diagonal to the parallelogram under the sine and cosine of F, or I; and its center B, or b, is thus also the center for the concyclic points $FfS'e$, or $IHS'K$, as the case may be; and therefore, again, is a common extremity of the minor axi [Theorem 3].

The interfocal distance $S'f$, or $S'K$, must therefore ever equal the minor axis of its anacyclic ellipse; since points S' and B, or b, are common; while if C and C', or C'' and C''', represent the respective centers of the two anacyclic curves; $S'C=BC'$, and $S'C'=BC$, or $S'C''=bC'''$, and $S'C'''=bC''$.

And so again; if X be taken to represent the magnitude $S'F$, or $S'D$, the radius of the common director circle, equalling the major axis; and Y be taken to represent the interfocal magnitude $S'f$ of the given ellipse; then the eccentric

ratios of the said ellipse, and of its reciprocal hyperbola will be represented by $Y:X$, and $X:Y$ respectively [Theorem 7]. While the eccentricity of the anacyclic hyperbola, as a conjugate in ratio, will obviously be $X:\sqrt{(X^2-Y^2)}$; and thus the ratio of its reciprocal ellipse—the anacyclic to the first ellipse—will be $\sqrt{(X^2-Y^2)}:X$.

As the reciprocal fixed points then, say P and H [Figure 4], of ellipse and hyperbola move from the center and infinity respectively, towards the circumference and each other, along the major axis, their anacyclic fixed points, L and K, are moving simultaneously in a reverse direction along the minor axis, away from the circumference and each other, towards the center or infinity. Instantaneously transposing with each other, and returning from thence back to the circumference; as the two parent points meet and cross on the circumference and pass onwards towards infinity and the center. The common determining radius—as *e. g.* $S'I$—during these cycles, swinging from the minor to the major axis, and back again. The diagonal—as *e. g.* HK—and the tangent and cotangent—as *e. g.* PIL—meanwhile rocking from a position, on the minor axis, at right angles to each other, through the medial parallels, corresponding to eBf and EFS, to a second position at right angles on the major axis; and then, reversing, back to their original position.

The medial forms, therefore, of both the hyperbolic and the elliptic cycles —lying midway between the parabola, the pivotal form of the greater reciprocal cycle, on the one hand; and its extremes, the circle and right line hyperbola at infinity, on the other—are alike determined by that radius $S'F$, which bisects in 45° the right angle between the axi. Giving us, then, outside of the director circle, two equilateral hyperbolas. And within it, two precisely similar ellipses, whose eccentric ratios, like those of their reciprocal equilateral hyperbolas, will be as $\sqrt{1}$ to $\sqrt{2}$, or $\sqrt{2}:\sqrt{1}$; so that the duplicate ratio of CS to CX, or CX to CS, in both alike, is as 1 to 2.

This medial ellipse, as we may call it, in default of a better term, has many important theorems in common with its reciprocal, the equilateral hyperbola; despite the fact that the elliptic, answering to the latter curve, is usually thought to be the circle. Yet a lack of space compels me to defer to a future occasion the discussion of some of these theorems; as well as others pertaining in general to "reciprocals" and "anacyclics."